Mathematics Framework Solution Sets
Algebra I

1

A note about these solutions.

These solutions are intended for teachers, not students. The solutions are fairly detailed and include additional comments that serve to further explain the content and purpose of each problem. It is important to note that the solutions are not meant to be representative of student solutions.

It is the nature of many mathematics problems that they can be solved in different ways. The solutions given here represent simply one way of solving the problems. At times alternative solution methods are mentioned in the "Further Explanation" boxes.

It is our hope that these solution sets help teachers to better see the essential skills and concepts that are important to student success in Algebra I.

> **1.0 Students identify and use the arithmetic properties of subsets of integers and rational, irrational, and real numbers, including closure properties for the four basic arithmetic operations where applicable:**
> **1.1 Students use properties of numbers to demonstrate whether assertions are true or false.**

Problem: Prove or give a counterexample: The average of rational numbers is a rational number.

> **Solution:** Let $r = \frac{a}{b}$ and $s = \frac{c}{d}$ be two arbitrary rational numbers. By definition $a, b, c,$ and d are integers, and b and d are not equal to 0. To find their average, we add them and divide by 2 as follows. First,
>
> $$\begin{aligned} r + s = \frac{a}{b} + \frac{c}{d} &= \frac{a}{b} \cdot 1 + \frac{c}{d} \cdot 1 \\ &= \frac{a}{b} \cdot \frac{d}{d} + \frac{c}{d} \cdot \frac{b}{b} \\ &= \frac{ad}{bd} + \frac{bc}{bd} \\ &= \frac{ad + bc}{bd}. \end{aligned}$$
>
> We have used the multiplicative identity property of 1 to find a common denominator in order to add the two fractions. Dividing by 2 is equivalent to multiplying by $\frac{1}{2}$, so the average becomes
>
> $$\text{average}(r, s) = \frac{1}{2} \cdot \frac{ad + bc}{bd} = \frac{ad + bc}{2bd}.$$
>
> Since the integers are closed under addition and multiplication, $ad + bc$ and $2bd$ are integers, and by the zero product property, since b and d are not equal to 0, $2bd \neq 0$. This shows that the average of two rational numbers is also a rational number.

Answer: The average of two rational numbers is a rational number.

> **Further Explanation:** The set of all rational numbers is denoted by $\mathbb{Q}$. A rational number by definition is a number that can be represented in the form p/q where p and q are integers and $q \neq 0$. In symbols:
>
> $$\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, \ q \neq 0\}.$$
>
> Notice in the solution above we have used the closure properties of the integers to show that the average of two rational numbers is also a rational number. Notice also that we have used the *zero product property* of the real numbers, that is,
>
> > If a and b are real numbers, then $a \cdot b = 0$ if and only if one (or both) of a or b is 0.

> **2.0 Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.**

Problem: I start with a number and apply a four-step process: (1) add 13; (2) multiply by 2; (3) take the square root; and (4) take the reciprocal. The result is 1/4. What number did I start with? If I start with the number x, write a formula that gives the result of the four-step process.

Solution: Let the starting number be represented by x. Then by the process described: (1) adding 13 yields $x + 13$; (2) multiplying the result by 2 yields $2(x + 13)$; (3) taking the square root gives $\sqrt{2(x + 13)}$; and finally (4) taking the reciprocal yields $1/\sqrt{2(x + 13)}$. Thus the formula for finding the number becomes:

$$\frac{1}{\sqrt{2(x + 13)}} = \frac{1}{4}.$$

By multiplying both sides by $4\sqrt{2(x + 13)}$ (or by taking the reciprocal of both sides), we get

$$\sqrt{2(x + 13)} = 4.$$

Square both sides to clear the radical:

$$2(x + 13) = 16,$$

which means $x + 13 = 8$ after dividing both sides by 2. Subtracting 13 from both sides we get

$$x + 13 - 13 = 8 - 13 \quad \Rightarrow \quad x = -5.$$

Answer: The process yields the equation $\dfrac{1}{\sqrt{2(x + 13)}} = \dfrac{1}{4}$. Solving this equation gives $x = -5$.

> **2.0 Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.**

Problem: What must be true about a real number x if $x = \sqrt{x^2}$?

> **Solution:** There are three possibilities for a real number that will affect the outcome.
>
> 1. If $x = 0$, then $\sqrt{x^2} = \sqrt{0^2} = \sqrt{0} = 0$, so that $x = \sqrt{x^2}$ when $x = 0$.
>
> 2. If $x > 0$, then $\sqrt{x^2} = x$.
>
> 3. If $x < 0$, then $x = -n$ for some positive real number n. Thus
>
> $$\sqrt{x^2} = \sqrt{(-n)^2} = \sqrt{n^2} = n = x,$$
>
> therefore when $x < 0$, $\sqrt{x^2} = x$.
>
> We see that $\sqrt{x^2} = x$ only when $x \geq 0$.

Answer: If $x = \sqrt{x^2}$, then x must be nonnegative.

> **Further Explanation:** It is important to remember that the symbol "$\sqrt{}$" denotes the *positive square root* of a number (with the exclusion of $\sqrt{0} = 0$). Thus, for example $\sqrt{36} = 6$. In the solution above we see then why $\sqrt{6^2} = \sqrt{36} = 6$, while $\sqrt{(-6)^2} = \sqrt{36} = 6$, so that $\sqrt{(-6)^2} = -6$.

> **2.0 Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.**

Problem: Write as a power of x: $\dfrac{\sqrt{x}}{x\sqrt[3]{x}}$.

Solution: We first rewrite the expression using fractional powers:

$$\frac{\sqrt{x}}{x\sqrt[3]{x}} = \frac{x^{\frac{1}{2}}}{x \cdot x^{\frac{1}{3}}}.$$

Using the additive law of exponents, we have $x \cdot x^{\frac{1}{3}} = x^1 \cdot x^{\frac{1}{3}} = x^{\frac{4}{3}}$. Therefore the expression becomes

$$\frac{x^{\frac{1}{2}}}{x \cdot x^{\frac{1}{3}}} = \frac{x^{\frac{1}{2}}}{x^{\frac{4}{3}}}.$$

Applying the law for dividing powers to the same base, we have

$$\frac{x^{\frac{1}{2}}}{x^{\frac{4}{3}}} = x^{\frac{1}{2}-\frac{4}{3}} = x^{-\frac{5}{6}}.$$

Answer: The expression is equivalent to $x^{-\frac{5}{6}}$.

2.0 Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

Problem: Solve for x: $|x^3| = \dfrac{1}{2\sqrt{2}}$.

Solution: First, notice that $2\sqrt{2} = 2 \cdot 2^{\frac{1}{2}} = 2^{\frac{3}{2}}$. By definition of negative exponents,

$$\frac{1}{2\sqrt{2}} = \frac{1}{2^{\frac{3}{2}}} = 2^{-\frac{3}{2}}.$$

Thus, the question is equivalent to solving $|x^3| = 2^{-\frac{3}{2}}$. By definition of the absolute value, we see that

$$x^3 = \pm 2^{-\frac{3}{2}}.$$

Recalling that $(x^3)^{\frac{1}{3}} = x^1$, we raise both sides of the equation to the 1/3 power, giving us

$$x = \left(\pm 2^{-\frac{3}{2}} \right)^{\frac{1}{3}}.$$

Recall that

$$(-a)^{\frac{1}{3}} = \sqrt[3]{-a} = -\sqrt[3]{a} = -a^{\frac{1}{3}}.$$

This allows us to pull the $\pm$ sign out and apply the rules of exponents to get

$$x = \pm 2^{-\frac{3}{2} \cdot \frac{1}{3}} = \pm 2^{-\frac{1}{2}}.$$

This means that

$$x = \pm \frac{1}{\sqrt{2}}.$$

Answer: The solution is $x = \pm\dfrac{1}{\sqrt{2}} = \pm\dfrac{\sqrt{2}}{2}$.

3.0 Students solve equations and inequalities involving absolute values.

Problem: Solve for x: $3|x| + 2 = 14$.

Solution: We isolate the absolute value on the left by first subtracting 2 and then dividing by 3 on both sides of the equation:

$$3|x| + 2 = 14$$
$$3|x| + 2 - 2 = 14 - 2$$
$$3|x| = 12$$
$$|x| = 4.$$

Therefore $x = 4$ or $x = -4$.

Answer: The solution set is $\{4, -4\}$.

Further Explanation: Recall that the absolute value of a number represents its distance from 0 on a number line. When x is positive, then this distance is simply x units, so $|x| = x$. If $x = 0$, then $|x| = 0$. However, if $x < 0$, then x is the same distance from 0 as its additive inverse, $-x$ (which is positive). Therefore $|x| = -x$ if $x < 0$. In general

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

It is helpful to think about the minus sign in the preceding formula for the absolute value of x as "the opposite of", as opposed to "negative".

3.0 Students solve equations and inequalities involving absolute values.

Problem: Express the solution using interval notation: $|x + 1| \geq 2$.

Solution: The inequality $|x + 1| \geq 2$ can be written as two inequalities using the definition of absolute value. If x is a number such that the quantity $x + 1$ is positive or 0, then $|x + 1| = x + 1$. On the other hand, if x is such that $x + 1$ is negative, then $|x + 1| = -(x + 1)$. This yields the two equations:

$$x + 1 \geq 2 \quad \text{and} \quad -(x + 1) \geq 2.$$

The first inequality yields $x \geq 1$, or the interval $[1, \infty)$. To solve the second inequality, we can multiply both sides of the inequality by -1, remembering to change the direction of the sign:

$$-(x + 1) \geq 2 \quad \Rightarrow \quad x + 1 \leq -2.$$

This inequality yields $x \leq -3$, or the interval $(-\infty, -3]$.

Answer: The solution set consists of all numbers less than or equal to -3 and all numbers greater than or equal to 1.

Further Explanation: Note that one may also solve the problem by utilizing the interpretation of the expression $|x - y|$ as the distance between x and y on a number line. If we rewrite the problem as $|x - (-1)| \geq 2$, then the problem becomes equivalent to asking "which numbers are at a distance of at least 2 from -1?" A graph of the solution set illustrates this idea:

$$|x - (-1)| \geq 2$$

Numbers here are at least
2 units from –1.
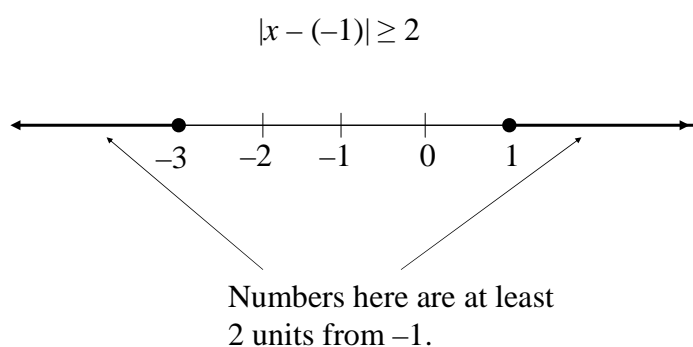
3.0 Students solve equations and inequalities involving absolute values.

Problem: Sketch the interval in the real number line that is the solution for $|x-5| < 2$.
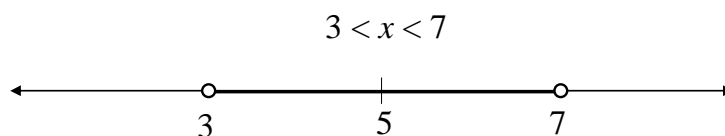
Solution: As in the previous problem, we can write $|x - 5| < 2$ as two inequalities:

$$x - 5 < 2 \quad \text{and} \quad -(x - 5) < 2,$$

which results in the two statements

$$x - 5 < 2 \quad \text{and} \quad x - 5 > -2.$$

These yield the two intervals $x < 7$ and $x > 3$. Notice that if x is to satisfy the original inequality, then x must lie in both of these intervals (that is, in their intersection). The solution set is $3 < x < 7$.

$$3 < x < 7$$

Answer: The solution set consists of all real numbers between 3 and 7, not including the endpoints.

Further Explanation: Again, one may solve this using the distance interpretation of $|x - y|$. In that case, the problem simply asks "which numbers are at a distance of no more than 2 from 5?" It is clear in the illustration above that any number greater than 3 but less than 7 will work.

> **4.0 Students simplify expressions before solving linear equations and inequalities in one variable, such as** $3(2x - 5) + 4(x - 2) = 12$**.**

Problem: Expand out and simplify $2(3x + 1) - 8x$.

> **Solution:** Using the distributive property:
>
> $$2(3x + 1) - 8x = 6x + 2 - 8x.$$
>
> Now combine like terms using the distributive property once more:
>
> $$6x + -8x + 2 = (6 + -8)x + 2$$
> $$= -2x + 2.$$

Answer: The simplified expression is $-2x + 2$.

4.0 Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x - 5) + 4(x - 2) = 12$**.**

Problem: Solve for x: $5x - 2 \leq -3(x + 1) + 2$.

Solution: Apply the distributive property to $-3(x + 1)$ first, to obtain

$$5x - 2 \leq -3x - 3 + 2.$$

Combine any remaining like terms on the same side of the inequality:

$$5x - 2 \leq -3x - 1.$$

To isolate x on one side of the inequality, we add 2 and $3x$ to either side, and simplify, obtaining

$$5x - 2 + 2 + 3x \leq -3x - 1 + 2 + 3x$$
$$5x + 3x \leq 1$$
$$8x \leq 1.$$

The solution set is $x \leq 1/8$.

Answer: The solution set is $x \leq 1/8$, or $(-\infty, 1/8]$.

Further Explanation: One may also isolate the variable on the right, to obtain

$$-1 \leq -8x.$$

To solve this simple inequality, we must remember to change the direction of the inequality if we divide by a negative number. We obtain the equivalent statement

$$1/8 \geq x.$$

Algebra I

> **4.0 Students simplify expressions before solving linear equations and inequalities in one variable, such as** $3(2x - 5) + 4(x - 2) = 12$**.**

Problem: Solve for x: $2 - (2 - (3x + 1) + 1) = 3(x - 2) + x$.

Solution: Use the distributive property on each side of the equation to eliminate parentheses, starting from the inside and moving out:

$$
\begin{aligned}
2 - (2 - (3x + 1) + 1) &= 3(x - 2) + x \\
2 - (2 - 3x - 1 + 1) &= 3x - 6 + x \\
2 - (2 - 3x) &= 4x - 6 \\
2 - 2 + 3x &= 4x - 6 \\
3x &= 4x - 6.
\end{aligned}
$$

Finally, subtracting $4x$ from both sides of the equation yields $-x = -6$, or $x = 6$.

Answer: The solution is $x = 6$.

4.0 Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x - 5) + 4(x - 2) = 12$**.**

Problem: Solve for x: $\dfrac{3}{x-2} = \dfrac{4}{x+5}$.

Solution: Multiply both sides of the equation by $(x-2)(x+5)$ to clear the denominators

$$\frac{3}{x-2} \cdot (x-2)(x+5) = \frac{4}{x+5}(x-2)(x+5).$$

Simplifying, we obtain

$$3(x + 15) = 4(x - 2) \quad \Rightarrow \quad 3x + 15 = 4x - 8.$$

This simplifies to $23 = x$.

Answer: The solution is $x = 23$.

> **5.0 Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.**

Problem: To compute the deduction that you can take on your federal tax return for medical expenses, you must deduct 7.5% of your adjusted gross income from your actual medical expenses. If your actual medical expenses are $1,600 and your deduction is less than $100, what can you conclude about your adjusted gross income? (CERT 1997, 22.)

Solution: Let I denote your adjusted gross income. Then 7.5% of your adjusted gross income is $7.5\% \times I = 0.075I$. Deducting 7.5% of your gross income from your medical expenses is equivalent to writing the expression $1600 - 0.075I$. Finally, the problem states that this quantity is less than $100, yielding the inequality

$$1600 - 0.075I < 100.$$

This is equivalent to

$$1500 < 0.075I \quad \Rightarrow \quad \frac{1500}{.075} < I,$$

or $I > 20,000$. Therefore, your gross adjusted income must be greater than $20,000.

Answer: Your adjusted gross income is greater than $20,000.

> **5.0 Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.**

Problem: Joe is asked to pick a number less than 100, and Moe is asked to guess it. Joe picks 63. Write an inequality that says that Moe's guess is within 15, inclusive, of the number Joe has in mind. Solve this inequality to find the range of possibilities for Moe's guess.

> **Solution:** Let M denote Moe's number. If Moe's number is within 15 of Joe's number, which is 63, then M is no less than $63 - 15$ and no greater than $63 + 15$. That is
> $$63 - 15 \leq M \leq 63 + 15 \quad \Rightarrow \quad 48 \leq M \leq 78.$$

Answer: The number is between 48 and 78 inclusive.

> **Further Explanation:** Equivalently, to solve this problem, we can use the expression $|M - 63|$, which denotes the difference between Moe's number M, and Joe's number 63. In that case, we'd have
>
> $$|M - 63| \leq 15.$$
>
> One should check that this is equivalent to $48 \leq M \leq 78$.

> **5.0 Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.**

Problem: Four more than three-fifths of a number is 24. Find the number.

> **Solution:** Let n represent the unknown number. Then three-fifths of the number is represented by $\frac{3}{5}n$, so that four more than three-fifths of the number is $\frac{3}{5}n + 4$. As stated in the problem,
>
> $$\frac{3}{5}n + 4 = 24 \quad \Rightarrow \quad \frac{3}{5}n = 20.$$
>
> Multiplying both sides of the equation by $\frac{5}{3}$, the multiplicative inverse of $\frac{3}{5}$, yields
>
> $$\frac{5}{3} \times \frac{3}{5}n = \frac{5}{3} \times 20 \quad \Rightarrow \quad 1 \times n = \frac{5}{3} \times 20,$$
>
> or
>
> $$n = \frac{5 \times 20}{3} = \frac{100}{3} = 33\frac{1}{3}.$$

Answer: The number is $33\frac{1}{3}$.

5.0 Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

Problem: Luis was thinking of a number. If he multiplied the number by 7, subtracted 11, added 5 times the original number, added -3, and then subtracted twice the original number, the result was 36. Use this information to write an equation that the number satisfies and then solve the equation.

Solution: Let n represent the number Luis was thinking of. Following the steps of his process yields the following equation:

$$7n - 11 + 5n + -3 - 2n = 36.$$

Simplify the left-hand side of the equation by combining like terms to obtain:

$$10n - 14 = 36.$$

Adding 14 to both sides of the equation yields $10n = 50$, which means $n = 5$.

Answer: The number is 5.

> **6.0 Students graph a linear equation and compute the x- and y-intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequalities (e.g., they sketch the region defined by $2x + 6y < 4$)**

Problem: The cost of a party at a local club is \$875 for 20 people and \$1,100 for 30 people. Assume that the cost is a linear function of the number of people. Write an equation for this function. Sketch its graph. How much would a party for 26 people cost? Explain and interpret the slope term in your equation. (CERT 1999, 61.)

Solution: The cost of the party depends on the number of people who attend. Let x represent the number of people at the party, and let y represent the cost of the party. We are given two points that satisfy the linear equation that represents the cost of the party:

$$(x_1, y_1) = (20, 875) \quad \text{and} \quad (x_2, y_2) = (30, 1100).$$

We'll use the two point form of the equation of a line:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \qquad (*),$$

and substitute the appropriate values from the ordered pairs given. The right hand side of the equation above represents the slope:

$$\frac{1100 - 875}{30 - 20} = \frac{225}{10} = \frac{45}{2} = \frac{22.5}{1}.$$

Expressing the slope as $\frac{22.5}{1}$, we see that it costs \$22.50 per person to attend the party. Using our points in equation $(*)$ above, we get:

$$\frac{y - 875}{x - 20} = 22.5.$$

To graph this equation, we'll put the equation above into slope-intercept form:

$$\frac{y - 875}{x - 20} = 22.5$$
$$y - 875 = (x - 20) \times 22.5$$
$$y - 875 = 22.5x - 450$$
$$y = 22.5x + 425.$$

When $x = 0$, there are no people at the party, and the cost of the club is \$425 (this is the y-intercept). To graph the equation of the line, let the x-axis represent the number of people at the party, and the y-axis represent the cost of the club rental. Then we see that for every 2 people that are added to the party, the cost increases by \$45. The graph is shown below:
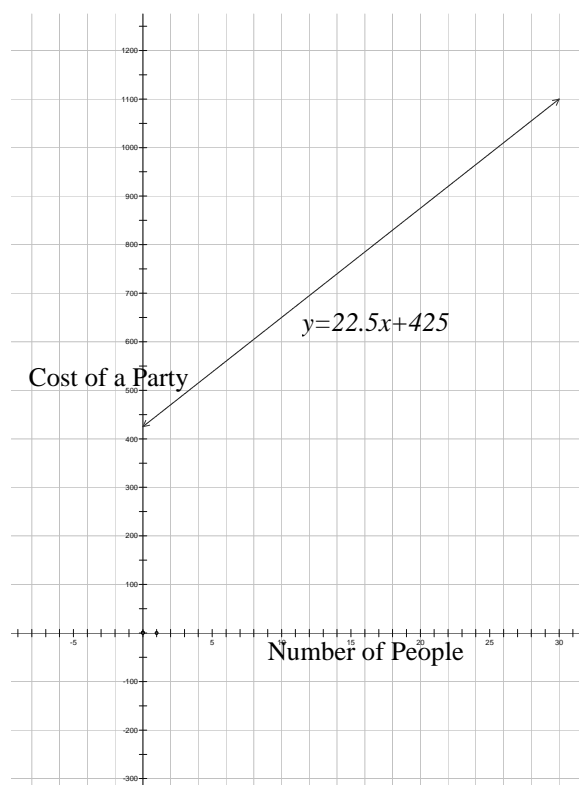
To find the cost for 26 people to attend the party, substitute $x = 26$. Then

$$y = 22.5(26) + 425 = 585 + 425 = 1010.$$

This means that the party would cost \$1010 for 26 people to attend.

Answer: The equation of the line is $y = 22.5x + 425$. The slope represents the cost per person, or \$22.50 per person. The cost for 26 people is \$1010.

Further Explanation: One may also use the *slope formula*, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, to first find the slope, and then use this to find the equation of the line, either by using the *slope-intercept formula*:
$$y = mx + b,$$
or the *point-slope formula*:
$$y - y_1 = m(x - x_1),$$
along with one of the given points.

6.0 Students graph a linear equation and compute the x- and y-intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequalities (e.g., they sketch the region defined by $2x + 6y < 4$)

Problem: Graph $2x - 3y = 4$. Where does the line intersect the x-axis? Where does the line intersect the y-axis? What is the slope?

Solution: Since the equation is in standard form, we can set $y = 0$ to find the x-intercept and then set $x = 0$ to find the y-intercept. When $y = 0$ we have
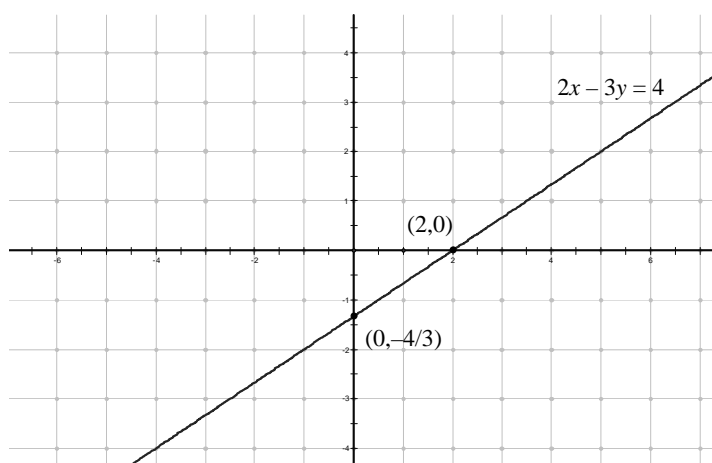
$$2x - 3(0) = 4 \quad \Rightarrow \quad 2x = 4,$$

so that $x = 2$. Thus the line intersects the x-axis at the point $(2, 0)$. When $x = 0$ we have

$$2(0) - 3y = 4 \quad \Rightarrow \quad -3y = 4,$$

so that $y = -4/3$. Thus the line intersects the y-axis at $(0, -4/3)$. The slope can easily be determined from the intercepts:

$$m = \frac{-\frac{4}{3} - 0}{0 - 2} = \frac{-\frac{4}{3}}{-2} = -\frac{4}{3} \cdot \frac{1}{-2} = \frac{2}{3}.$$

To graph the line, draw the unique straight line through the intercepts:


Answer: The slope is $2/3$, while the x- and y-intercepts are $(2, 0)$ and $(0, -4/3)$ respectively.

6.0 Students graph a linear equation and compute the x- and y-intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequalities (e.g., they sketch the region defined by $2x + 6y < 4$)

Problem: Sketch the region in the x-y plane that satisfies both of the following inequalities: $y < 3x + 1$, $2x + 3y + 8 > 0$.

Solution: It is useful to rewrite the second inequality as $y > -\frac{2}{3}x - \frac{8}{3}$. To find the region, we first graph the linear equations that determine the boundaries: $y = 3x + 1$ and $y = -\frac{2}{3}x - \frac{8}{3}$. The first linear equation, $y = 3x + 1$, determines a line of slope 3 with y-intercept $(0, 1)$. The second linear equation $y = -\frac{2}{3}x - \frac{8}{3}$ determines a line of slope $-2/3$ with y-intercept $(0, -8/3)$. They are graphed on the same set of axes below:
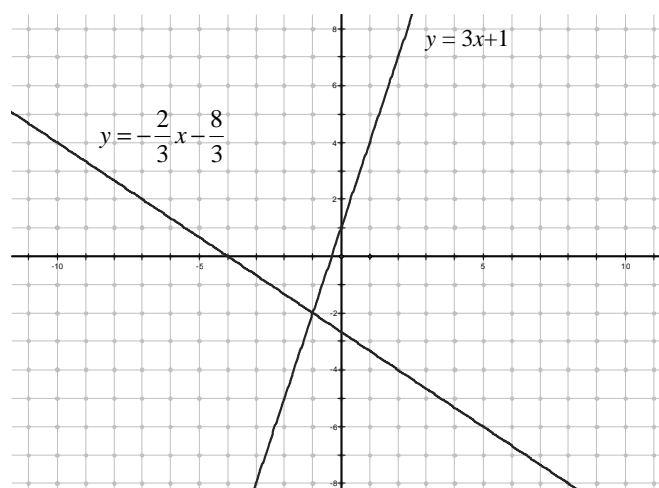
In the graph of the solution the lines will be dotted. This is because the inequalities are *strict* inequalities so that points on the lines are not included in the solution set. By testing points, we see that the inequality $y < 3x + 1$ describes the region below the line $y = 3x + 1$. Also, the inequality $y > -\frac{2}{3}x - \frac{8}{3}$ describes the region above (or to the right of) the line $y = -\frac{2}{3}x - \frac{8}{3}$. (*continued on next page.*)

The solution to the *system* of inequalities is the region in common to both, it is represented by the shaded area in the graph below:
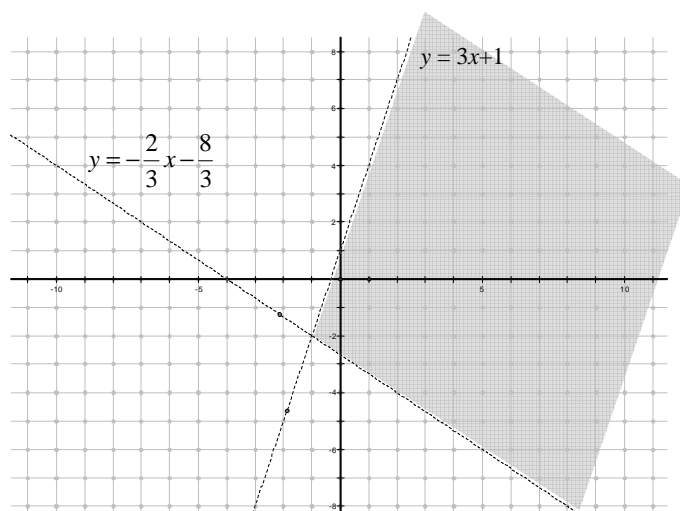
Answer: The solution set is graphed above.

Further Explanation: Notice that the point $(0,3)$ satisfies $y > -\frac{2}{3}x - \frac{8}{3}$, but not $y < 3x+1$. Therefore, the upper of the four regions cannot be included in the solution set. Similarly, since the point $(0,-4)$ satisfies only $y < 3x+1$, the solution set omits this region. Finally, since $(0,0)$ satisfies both inequalities, the region which contains it must be included in the solution set.

> **6.0 Students graph a linear equation and compute the x- and y-intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequalities (e.g., they sketch the region defined by $2x + 6y < 4$).**
> **7.0 Students verify that a point lies on a line, given an equation of the line. Students are able to derive linear equations by using the point-slope formula.**

Problem: Find an equation for the line that passes through $(2, 5)$ and $(-3, 1)$. Where does the line intersect the x-axis? Where does the line intersect the y-axis? What is the slope?

> **Solution:** In this case, we'll use the slope formula
>
> $$m = \frac{y_2 - y_1}{x_2 - x_1},$$
>
> with the given points to determine the slope. To that end, we have
>
> $$m = \frac{1 - 5}{-3 - 2} = \frac{-4}{-5} = \frac{4}{5}.$$
>
> Using the point-slope form of the equation of a line with the given point $(2, 5)$, we obtain
>
> $$y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - 5 = \frac{4}{5}(x - 2).$$
>
> Multiplying each side by 5 will clear the denominator, and distributing allows us to simplify. We obtain
>
> $$5y - 25 = 4x - 8 \quad \Rightarrow \quad 0 = 4x - 5y + 17.$$
>
> This is the standard form of the equation of a line. To find the x-intercept, set $y = 0$ and solve for x, obtaining $x = -\frac{17}{4}$. To find the y-intercept, set $x = 0$ and solve for y, obtaining $y = \frac{17}{5}$.

Answer: The equation of the line is $4x - 5y + 17 = 0$. The slope is $4/5$. The points $(-\frac{17}{4}, 0)$ and $(0, \frac{17}{5})$ are the x- and y-intercepts, respectively.

> **Further Explanation:** When we arrived at $y - 5 = \frac{4}{5}(x - 2)$ we could also solve for y to convert the equation into the equivalent slope-intercept form. In that case, we have
>
> $$y = \frac{4}{5}x - \frac{8}{5} + 5 \quad \text{or} \quad y = \frac{4}{5}x + \frac{17}{5}.$$
>
> To find the intercepts, first notice that $\frac{17}{5}$ is the y-intercept. Next, set $y = 0$ and solve for x to find the x-intercept.

> **6.0 Students graph a linear equation and compute the x- and y-intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequalities (e.g., they sketch the region defined by $2x + 6y < 4$)**
> **7.0 Students verify that a point lies on a line, given an equation of the line. Students are able to derive linear equations by using the point-slope formula.**

Problem: Find an equation for the line that passes through $(5, 3)$ and $(5, -2)$. Where does the line intersect the x-axis? Where does the line intersect the y-axis? What is the slope?

> **Solution:** Since the x-coordinates of the points are the same, the line must be a vertical line intersecting the x-axis at $(5, 0)$. In that case, we say the slope is undefined, or that the line has no slope. The line does not cross the y-axis; it is parallel to the y-axis.

Answer: The equation of the line is $x = 5$, indicating that any point with an x-coordinate equal to 5 will lie on the line. It is a vertical line, so it has no slope (equivalently, the slope is undefined). It crosses the x-axis at $(5, 0)$ but does not cross the y-axis.

> **Further Explanation:** If we try to find the slope of the line, we obtain
>
> $$m = \frac{3 - (-2)}{5 - 5},$$
>
> which is undefined as the denominator will be 0. Alternatively, if we substitute each point into the slope-intercept equation of a line ($y = mx + b$), we obtain
>
> $$3 = 5m + b \quad \text{and} \quad -2 = 5m + b.$$
>
> Since the right hand sides of these equations are equal, this would imply that $3 = -2$, which is impossible.

7.0 Students verify that a point lies on a line, given an equation of the line. Students are able to derive linear equations by using the point-slope formula.

Problem: The weight of a pitcher of water is a linear function of the depth of the water in the pitcher. When there are 2 inches of water in the pitcher, it weighs 2 lbs.; and when there are 8 inches of water in the pitcher, it weighs 5 lbs. Find a formula for the weight of the pitcher as a function of the depth of the water.

Solution: Let x represent the depth of the water in inches and y the weight in pounds. Assuming the relationship is linear, we find the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 2}{8 - 2} = \frac{3}{6} = \frac{1}{2}.$$

Using the point-slope form of the equation of a line with one of our given points, we obtain

$$y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - 5 = \frac{1}{2}(x - 8),$$

or

$$y = \frac{1}{2}x + 1.$$

Answer: The weight of the water can be represented as a function of the depth with the equation $y = \frac{1}{2}x + 1$.

7.0 Students verify that a point lies on a line, given an equation of the line. Students are able to derive linear equations by using the point-slope formula.

Problem: Find an equation for the line that passes through $(-2, 5)$ and has slope $-2/3$.

Solution: Using the point-slope form of the equation of a line:

$$y - 5 = -\frac{2}{3}(x - (-2)) \quad \Rightarrow \quad y - 5 = -\frac{2}{3}x - \frac{4}{3}.$$

Adding 5 to both sides, we get

$$y = -\frac{2}{3}x - \frac{4}{3} + 5 \quad \Rightarrow \quad y = -\frac{2}{3}x + \frac{11}{3}.$$

Answer: The equation of the line is $y = -\frac{2}{3}x + \frac{11}{3}$.

8.0 Students understand the concepts of parallel lines and perpendicular lines and how their slopes are related. Students are able to find the equation of a line perpendicular to a given line that passes through a given point.

Problem: Find the equation of the line that is perpendicular to the line through $(2, 7)$ and $(-1, 3)$ and passes through the x-intercept of that line.

Solution: The line through $(2, 7)$ and $(-1, 3)$ has slope $m = \dfrac{7-3}{2-(-1)} = \dfrac{4}{3}$. To find the equation of the line, we solve for y in the equation:

$$\frac{y-7}{x-2} = \frac{4}{3}.$$

Thus,

$$3(y-7) = 4(x-2)$$
$$3y - 21 = 4x - 8$$
$$3y = 4x + 13$$
$$y = \frac{4}{3}x + \frac{13}{3}.$$

Setting $y = 0$, we obtain

$$0 = \frac{4}{3}x + \frac{13}{3},$$

or

$$-\frac{4}{3}x = \frac{13}{3}$$
$$-4x = 13$$
$$x = -\frac{13}{4}.$$

This gives us the x-intercept of the original line as $\left(-\dfrac{13}{4}, 0\right)$. To find the perpendicular line, recall that perpendicular lines have slopes that are negative reciprocals of each other. Therefore, the line in question has the form

$$y = -\frac{3}{4}x + b.$$

(*continued on next page.*)

We can find the y-intercept b by substituting the point $\left(-\dfrac{13}{4}, 0\right)$ into the equation and solving for b:

$$0 = \left(-\frac{3}{4}\right) \cdot \left(-\frac{13}{4}\right) + b$$
$$0 = \frac{39}{16} + b$$
$$b = -\frac{39}{16}.$$

Therefore, the line we are looking for is

$$y = -\frac{3}{4}x - \frac{39}{16}.$$

Answer: The line is $y = -\dfrac{3}{4}x - \dfrac{39}{16}$.

Further Explanation: Often, the relationship between the slopes of perpendicular lines ℓ_1 and ℓ_2 with non-zero slopes is expressed in the following way:

$$m_1 \cdot m_2 = -1,$$

where m_1 and m_2 are the slopes of ℓ_1 and ℓ_2 respectively. Of course, this means that if the slope of ℓ_1 is $m = 0$, then the slope of a line ℓ_2 perpendicular to ℓ_1 is $-1/m$.

8.0 Students understand the concepts of parallel lines and perpendicular lines and how their slopes are related. Students are able to find the equation of a line perpendicular to a given line that passes through a given point.

Problem: Are the following two lines perpendicular, parallel, or neither?

$$2x + 3y = 5$$
$$3x + 2y - 1 = 0.$$

Solution: First, we write the equations in slope-intercept form. For the first equation:

$$2x + 3y = 5$$
$$3y = -2x + 5$$
$$y = -\frac{2}{3}x + \frac{5}{3}$$

Then, for the second:

$$3x + 2y - 1 = 0$$
$$2y = -3x + 1$$
$$y = -\frac{3}{2}x + \frac{1}{2}.$$

Since the lines do not have the same slope, they are not parallel, hence they intersect. However, since the lines do not have slopes that are negative reciprocals of each other, they are not perpendicular. Thus they are neither.

Answer: The lines are neither parallel nor perpendicular.

Further Explanation: Two lines $\ell_1 : y = m_1 x + b_1$ and $\ell_2 : y = m_2 x + b_2$ are parallel if and only if their slopes are equal while their y-intercepts are not equal, that is $m_1 = m_2$ while $b_1 = b_2$. All vertical lines (undefined slopes) are parallel.

8.0 Students understand the concepts of parallel lines and perpendicular lines and how their slopes are related. Students are able to find the equation of a line perpendicular to a given line that passes through a given point.

Problem: If the line through $(1, 3)$ and $(a, 9)$ is parallel to $3x - 5y = 2$, then what is a?

Solution: We find the slope of the line $3x - 5y = 2$:

$$3x - 5y = 2$$
$$-5y = -3x + 2$$
$$y = \frac{3}{5}x - \frac{2}{5}.$$

If the line through the points $(1, 3)$ and $(a, 9)$ is parallel to this line, then the slope between these two points must be $\frac{3}{5}$ as well. Thus

$$\frac{9 - 3}{a - 1} = \frac{3}{5}$$
$$\frac{6}{a - 1} = \frac{3}{5}$$
$$30 = 3(a - 1)$$
$$10 = a - 1.$$

Therefore, $a = 11$.

Answer: The value of a is 11.

9.0 Students solve a system of two linear equations in two variables alge-
braically and are able to interpret the answer graphically. Students are
able to solve a system of two linear inequalities in two variables and to
sketch the solution sets.

Problem: Line 1 has equation $3x + 2y = 3$, and line 2 has equation $-2x + y = 5$.
Find the point of intersection of the two lines.

Solution: We solve the system simultaneously, by eliminating the coefficient of y.
To that end, we multiply each side of the equation defining line 2 by -2:

$$-2x + y = 5 \quad \xrightarrow{\times(-2)} \quad 4x - 2y = -10.$$

Then we combine this equation with the equation for line 1:

$$3x + 2y = 3$$
$$\underline{4x - 2y = -10}$$
$$7x + 0y = -7.$$

Solving for x gives $x = -1$. We can substitute $x = -1$ into either equation to obtain
the y-coordinate of the intersection point. Using the equation for line 1:

$$3(-1) + 2y = 3 \quad \Rightarrow \quad 2y = 6.$$

Therefore, $y = 3$, and the point of intersection is $(-1, 3)$.

Answer: The lines intersect at the point $(-1, 3)$.

9.0 Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

Problem: Sketch a graph of the values of x and y that satisfy both of the inequalities: $3x + 2y \geq 3$, $-2x + y \leq 5$.

Solution: We start by graphing the lines that border the region, which are given by the equations $3x + 2y = 3$ and $-2x + y = 5$. Notice that $(0, 3/2)$ and $(1, 0)$ are the intercepts of the first equation, while $(0, 5)$ and $(-5/2, 0)$ are the intercepts of the second. We can use these to plot the lines. Since the point $(0, 3)$ satisfies both inequalities, we shade the region that includes the point $(0, 3)$. The parts of the lines that border this region are included in the solution since the inequalities are not strict inequalities.
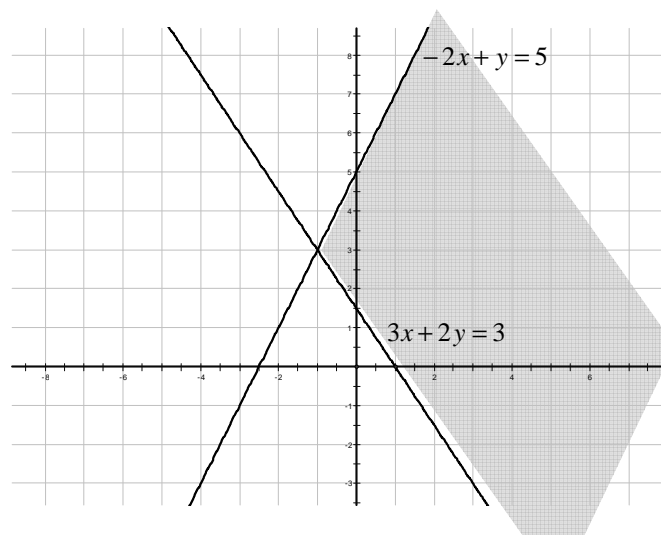

Answer: The solution set is graphed above.

10.0 Students add, subtract, multiply, and divide monomials and polynomials. Students solve multistep problems, including word problems, by using these techniques.

Problem: The volume of a rectangular prism with a triangular base is $36m^3 - 72m^2 + 29m - 3$. Assume that the height of the prism is $3m - 1$ and the height of the triangle is $6m - 1$. What is the base of the triangle?

Solution: The volume of the prism is $V = Bh$, where B represents the area of the triangular base and h represents the height of the prism. On the other hand, the area of the triangular base is $B = \frac{1}{2}bh_T$, where b is the length of the base of the triangle that defines the triangular base, and h_T is the height of this triangle. (See the figure for details.)
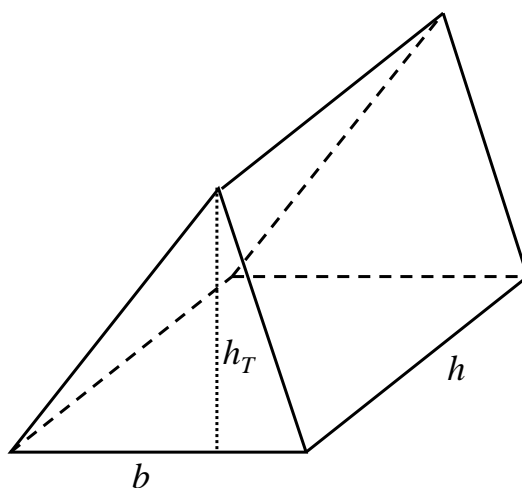
Thus, the formula for the volume becomes

$$V = Bh = \left(\frac{1}{2}bh_T\right)h.$$

We can rearrange the formula to find b in terms of the other variables:

$$V = \left(\frac{1}{2}bh_T\right)h$$
$$2V = bh_Th$$
$$b = \frac{2V}{h_Th}.$$

As the problem states, we have

$$b = \frac{2(36m^3 - 72m^2 + 29m - 3)}{(6m - 1)(3m - 1)}.$$

To simplify this, we perform polynomial long division:

$$
\begin{array}{r}
6m^2 - 11m + 3 \\
6m - 1 \overline{)\ 36m^3 - 72m^2 + 29m - 3} \\
\underline{-\ 36m^3\ \ +\ 6m^2} \\
-66m^2 + 29m \\
\underline{66m^2 - 11m} \\
18m - 3 \\
\underline{-\ 18m + 3} \\
0
\end{array}
$$

Thus we can rewrite the expression for b as

$$b = \frac{2(6m - 1)(6m^2 - 11m + 3)}{(6m - 1)(3m - 1)}.$$

We perform polynomial long division once more and obtain:

$$
\begin{array}{r}
2m - 3 \\
3m - 1 \overline{)\ 6m^2 - 11m + 3} \\
\underline{-\ 6m^2\ \ +\ 2m} \\
-9m + 3 \\
\underline{9m - 3} \\
0
\end{array}
$$

Thus,

$$b = \frac{2(6m - 1)(3m - 1)(2m - 3)}{(6m - 1)(3m - 1)}.$$

Reducing by noticing that $\dfrac{(6m - 1)}{(6m - 1)}$ and $\dfrac{(3m - 1)}{(3m - 1)}$ are both equal to 1, we get

$$b = 2(2m - 3) = 4m - 6.$$

Answer: The base of the triangle is given by the expression $b = 4m - 6$.

> **10.0 Students add, subtract, multiply, and divide monomials and polynomials. Students solve multistep problems, including word problems, by using these techniques.**

Problem: Simplify $[(3b^2 - 2b + 4) - (b^2 + 5b - 2)](b + 2)$.

> **Solution:** Working first in the brackets, we distribute (-1) to the expression in the second parenthesis:
>
> $$-(b^2 + 5b - 2) = (-1)(b^2 + 5b - 2) = -b^2 - 5b + 2.$$
>
> Thus, we have
>
> $$[(3b^2 - 2b + 4) - (b^2 + 5b - 2)](b + 2) = [3b^2 - 2b + 4 - b^2 - 5b + 2](b + 2)$$
> $$= [2b^2 - 7b + 6](b + 2).$$
>
> We apply the distributive property first with b, then with 2, and combine the resulting like terms:
>
> $$[2b^2 - 7b + 6](b + 2) = 2b^3 - 7b^2 + 6b + 4b^2 - 14b + 12$$
> $$= 2b^3 - 3b^2 - 8b + 12.$$

Answer: The simplified expression is $2b^3 - 3b^2 - 8b + 12$.

11.0 Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.

Problem: Solve for x: $\dfrac{x^2 - 4}{x - 2} + x^2 - 4 = 0$.

Solution: Since the linear expression $x - 2$ appears in the denominator of the fraction, $x = 2$ cannot be a solution to the problem. We factor the numerator of the first expression, and simplify:

$$\frac{x^2 - 4}{x - 2} + x^2 - 4 = 0$$

$$\frac{(x - 2)(x + 2)}{x - 2} + x^2 - 4 = 0$$

$$x + 2 + x^2 - 4 = 0 \qquad \text{since} \quad \frac{x - 2}{x - 2} = 1$$

$$x^2 + x - 2 = 0.$$

Now we can factor and apply the zero product property to find the solutions:

$$(x - 1)(x + 2) = 0 \quad \Rightarrow \quad x - 1 = 0 \quad \text{or} \quad x + 2 = 0.$$

In the first case, $x = 1$; in the second, $x = -2$.

Answer: The solution set is $\{1, -2\}$.

Algebra I

> **12.0 Students simplify fractions with polynomials in the numerator and denominator by factoring both and reducing them to the lowest terms.**

Problem: Reduce to lowest terms: $\dfrac{x^3 + x^2 - 6x}{x^2 + 13x + 30}$.

Solution: We factor the numerator and the denominator:

$$x^3 + x^2 - 6x = x(x^2 + x - 6) = x(x - 2)(x + 3)$$
$$x^2 + 13x + 30 = (x + 10)(x + 3).$$

The expression then becomes

$$\frac{x^3 + x^2 - 6x}{x^2 + 13x + 30} = \frac{x(x - 2)(x + 3)}{(x + 10)(x + 3)}.$$

Since $\dfrac{(x + 3)}{(x + 3)} = 1$, the simplified expression is

$$\frac{x(x - 2)}{x + 10}.$$

Answer: The expression reduces to $\dfrac{x(x - 2)}{x + 10}$.

12.0 Students simplify fractions with polynomials in the numerator and denominator by factoring both and reducing them to the lowest terms.

Problem: Solve for x: $\dfrac{3}{x-1} + \dfrac{10}{x^2 - 2x + 1} = 4$.

Solution: If we factor the denominator of the second fraction, we obtain $x^2 - 2x + 1 = (x-1)^2$. (Notice therefore that $x = 1$ cannot be a solution to the equation.) Multiply each term of the equation by the common denominator, which is $(x-1)^2$, and simplify:

$$\frac{3}{x-1} + \frac{10}{(x-1)^2} = 4$$

$$\frac{3}{x-1}\ (x-1)^2 + \frac{10}{(x-1)^2}\ (x-1)^2 = 4(x-1)^2$$

$$3(x-1) + 10 = 4(x-1)^2.$$

We have divided out common factors in numerators and denominators. Simplify further to obtain

$$3x - 3 + 10 = 4(x^2 - 2x + 1) \quad \Rightarrow \quad 3x + 7 = 4x^2 - 8x + 4.$$

Adding $-(3x + 7)$ to each side of the equation yields

$$0 = 4x^2 - 11x - 3 \quad \Rightarrow \quad 0 = (4x + 1)(x - 3).$$

Applying the zero product property, we see that

$$4x + 1 = 0 \quad \text{or} \quad x - 3 = 0.$$

Thus, $x = -\dfrac{1}{4}$ or $x = 3$.

Answer: The solution set is $\{-1/4, 3\}$.

Further Explanation: Notice that one can also combine the two fractions on the left-hand side of the equation, as follows:

$$\frac{3}{x-1} + \frac{10}{(x-1)^2} = \frac{3}{x-1} \times \frac{(x-1)}{(x-1)} + \frac{10}{(x-1)^2}$$

$$= \frac{3(x-1)}{(x-1)^2} + \frac{10}{(x-1)^2}$$

$$= \frac{3x + 7}{(x-1)^2}.$$

One can then solve the problem by setting this expression equal to 4 and then continuing to solve for x.

13.0 Students add, subtract, multiply, and divide rational expressions and functions. Students solve both computationally and conceptually challenging problems by using these techniques.

Problem: Solve for x: $\dfrac{\dfrac{x+2}{x-3} \times \dfrac{x^2+5x-24}{x-6}}{x+8} + 3 = 0.$

Solution: Notice that -8, 3 and 6 cannot be solutions to the equation. Multiplying each term of the equation by $(x+8)$ will remove the denominator of the complex fraction, leaving us with

$$\frac{x+2}{x-3} \times \frac{x^2+5x-24}{x-6} + 3(x+8) = 0.$$

If we factor the expression $x^2 + 5x - 24$ as $(x+8)(x-3)$, then we can simplify the first expression as follows:

$$\begin{aligned}
\frac{x+2}{x-3} \times \frac{x^2+5x-24}{x-6} &= \frac{x+2}{x-3} \times \frac{(x+8)(x-3)}{x-6} \\
&= \frac{x+2}{1} \times \frac{x+8}{x-6} \\
&= \frac{(x+2)(x+8)}{x-6}
\end{aligned}$$

The equation then becomes

$$\frac{(x+2)(x+8)}{x-6} + 3(x+8) = 0.$$

If we multiply through by $(x-6)$, we obtain

$$(x+2)(x+8) + 3(x+8)(x-6) = 0.$$

Notice that each term has a common factor of $(x+8)$. Applying the distributive property then gives

$$\begin{aligned}
(x+2)\underline{(x+8)} + 3\underline{(x+8)}(x-6) &= 0 \\
\underline{(x+8)}[(x+2) + 3(x-6)] &= 0 \\
(x+8)[x+2+3x-18] &= 0 \\
(x+8)(4x-16) &= 0.
\end{aligned}$$

Applying the zero product property, we obtain $x = -8$ or $x = 4$. But as we noted earlier, $x = -8$ cannot be a solution, so the only solution is $x = 4$.

Answer: The solution is $x = 4$.

14.0 Students solve a quadratic equation by factoring or completing the square.

Problem: Where does the graph of $f(x) = \dfrac{x^3 + 2x^2 - 15x}{x + 1}$ intersect the x-axis?

Solution: The graph of a function $f(x)$ will intersect the x-axis when $y = f(x) = 0$. Therefore we set
$$f(x) = \frac{x^3 + 2x^2 - 15x}{x + 1} = 0.$$
The only time the expression for $f(x)$ is 0 is when the numerator is 0. Thus we can set the numerator equal to zero and solve:

$$x^3 + 2x^2 - 15x = 0$$
$$x(x^2 + 2x - 15) = 0 \qquad \text{factor out the common } x$$
$$x(x - 3)(x + 5) = 0.$$

This expression is zero if any of the factors is zero, hence the x-values for which $f(x) = 0$ are $x = 0$, 3, and -5.

Answer: The graph of $f(x)$ crosses the x-axis at the points $(-5, 0)$, $(0, 0)$ and $(3, 0)$.

Further Explanation: The graph of a real-valued function $f(x)$ is the set of all ordered pairs of real numbers (x, y) such that $y = f(x)$. That is

$$\text{graph}(f) = \{(x, f(x)) \mid x \in \mathbb{R}\}.$$

To find where the graph of $f(x)$ crosses the x-axis, we are looking for those points $(x, y) \in \text{graph}(f)$ such that $y = 0$, that is, such that $f(x) = 0$.

15.0 **Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.**

Problem: Mary drove to work on Thursday at 40 miles per hour (mph) and arrived one minute late. She left at the same time on Friday, drove at 45 mph, and arrived one minute early. How far does Mary drive to work? (CERT 1999, 31)

Solution: Let t represent the time it takes for Mary to arrive at work on time, in hours. In that case, $t + \frac{1}{60}$ represents the time it took her to arrive on Thursday when she was one minute late, while $t - \frac{1}{60}$ represents the the time it took her to arrive on Friday when she was one minute early. Since she drove at 40 mph on Thursday, and 45 mph on Friday, the expressions

$$40 \quad t + \frac{1}{60}$$

$$45 \quad t - \frac{1}{60}$$

both represent the distance Mary drives to work. If we set these expressions equal and solve for t, we will find the amount of time it would take for Mary to arrive at work on time.

$$40 \quad t + \frac{1}{60} \quad = 45 \quad t - \frac{1}{60}$$

$$40t + \frac{40}{60} = 45t - \frac{45}{60}$$

$$40t + \frac{2}{3} = 45t - \frac{3}{4}.$$

This implies that

$$5t = \frac{2}{3} + \frac{3}{4} = \frac{17}{12},$$

or that $t = \frac{17}{60}$.

This means it takes Mary 17 minutes to arrive at work on time. On Thursday it took her 18 minutes at 40 mph. Therefore, she must travel

$$40 \, \text{mph} \times \frac{18}{60} \, \text{hr} = \frac{2}{3} \cdot 18 \, \text{mi}$$

or 12 miles to work.

Answer: Mary drives 12 miles to work.

15.0 Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.

Problem: Suppose that peanuts cost $.40/lb. and cashews cost $.72/lb. How many pounds of each should be used to make an 80 lb. mixture that costs $.48/lb.?

Solution: Let P represent the number of pounds of peanuts in the mix, and C the number of pounds of cashews in the mix. Then since $80\,$lb of nuts were purchased, we get an equation (i) $P + C = 80$. The cost of the peanuts in the mixture is $.40P$, while the cost of the cashews is $.72C$. The total cost of the mixture is

$$\$.48/\,\mathrm{lb} \times 80\,\mathrm{lb} = \$38.40,$$

so we get another equation (ii) $.4P + .72C = 38.4$. Manipulating equation (i) yields $P = 80 - C$. If we substitute this into equation (ii), we obtain

$$.4(80 - C) + .72C = 38.4$$

Simplifying, we have

$$32 - .4C + .72C = 38.4 \quad \Rightarrow \quad .32C = 6.4$$

Clearly then $C = 20\,$lb, and since $P + C = 80$, $P = 60\,$lb.

Answer: The mixture is composed of $60\,$lb of peanuts and $20\,$lb of cashews.

16.0 Students understand the concepts of a relation and a function, determine whether a given relation defines a function, and give pertinent information about given relations and functions.

Problem: The following points lie on the graph of a relation between x and y: $(0,0)$, $(-2,3)$, $(3,-2)$, $(2,3)$, $(-3,-3)$, $(2,-2)$. Can y be a function of x? Explain. Can x be a function of y? Explain.

Solution: A relation will determine y as a function of x if for every value of x there is one (and only one) value of y it is paired with. Since $(2,3)$ and $(2,-2)$ are members of this relation, y is not a function of x.
Similarly, x is a function of y if for each value of y there is one (and only one) value of x it is paired with. Since $(3,-2)$ and $(2,-2)$ are members of this relation x is not a function of y.

Answer: This relation does not determine y as a function of x nor x as a function of y.

Further Explanation: A *relation* between non-empty sets A and B is a collection of ordered pairs (x,y) in which each $x \in A$ is paired with an element $y \in B$. The set A is called the *domain* of the relation, while B is called the *codomain*. For instance, any collection of ordered pairs of real numbers determines a relation for appropriately defined A and B. As an example, consider the unit circle

$$C = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}.$$

A *function f* from A to B is a relation between A and B such that each member in A is paired with *exactly one* element of B. In this case, if (x,y) is a member of the relation, we write $y = f(x)$.
In the case that A and B are subsets of $\mathbb{R}$, we have the aid of the Cartesian plane in visualizing a relation or function by graphing coordinate pairs (x,y).

17.0 Students determine the domain of independent variables and the range of dependent variables defined by a graph, a set of ordered pairs, or a symbolic expression.

Problem: Determine the domain of the function $f(x) = \sqrt{|x| - 6}$.

Solution: The function $f(x)$ can be viewed as a composition of functions, $f(x) = g(h(x))$, where $h(x) = |x| - 6$ and $g(x) = \sqrt{x}$. The function $g(x)$ is only defined when $x \geq 0$; therefore, $g(h(x))$ is only defined when $h(x) > 0$. In this case, we need to find the values of x so that
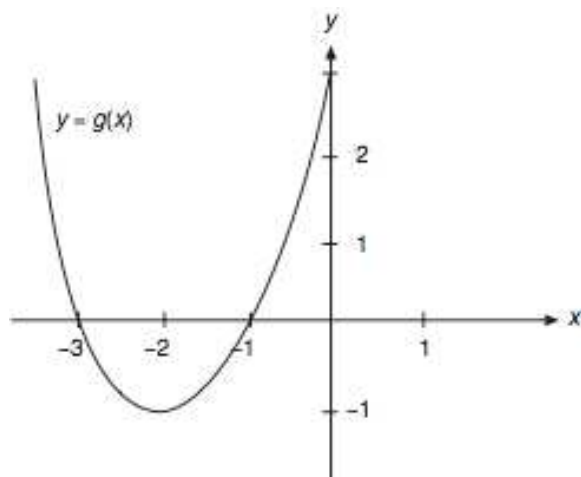$$h(x) = |x| - 6 \geq 0.$$
This is equivalent to $|x| \geq 6$. Thus either $x \geq 6$, or $-x \geq 6$, that is, $x \leq -6$.

Answer: The domain of $f(x)$ is $\{x \leq -6\} \cup \{x \geq 6\}$.

17.0 Students determine the domain of independent variables and the range of dependent variables defined by a graph, a set of ordered pairs, or a symbolic expression.

Problem: Determine the range of the function g whose graph is shown below.

Solution: Assume that the graph of g is a parabola and that the domain of g is all real numbers. In that case the point $(-2, -1)$ represents a minimum of the quadratic function g, that is, $g(-2) = -1$ is a minimum of g. Thus for any $x = -2$, g will take on a value greater than -1. This implies that the range of g is the set $\{y \geq -1\}$.

Answer: The range of g is $\{y \in \mathbb{R} \,|\, y \geq -1\}$.

> **17.0 Students determine the domain of independent variables and the range of dependent variables defined by a graph, a set of ordered pairs, or a symbolic expression.**

Problem: Let $f(x) = x^2 - 16$ (in words x squared minus 16, and x is a real number).

1. What is the domain of $f(x)$?

2. What is the range of $f(x)$?

3. For what values of x is $f(x)$ negative?

4. What are the domain and range of the square root of $(x^2 - 16)$ when x is assumed to be a real number?

Solution:

1. There are no restrictions on which values of x may be inputs for the function f. Thus, the domain of f is all real numbers.

2. This is a quadratic function of x. In vertex form, f looks like

$$f(x) = 1(x - 0)^2 + (-16),$$

which implies that $(0, -16)$ is the vertex of the parabola. In that case, the minimum of f occurs at 0: $f(0) = -16$. Thus, the range of f is $y \geq -16$.

3. Notice that $f(x) = 0$ when $x^2 = 16$, or when $x = 4$ or $x = -4$. Since $f(0) < 0$, $f(x)$ must be negative for x between -4 and 4.

4. Since the square root function is undefined for negative values, the domain of $\sqrt{x^2 - 16} = \sqrt{f(x)}$ must be the set of x-values for which $f(x) \geq 0$. This is the complement of the interval $-4 < x < 4$, which is $\{-4 \leq x\} \cup \{x \geq 4\}$. The range of the square root function is $\{y \geq 0\}$.

Answer: The domain of f consists of all real numbers. The range is $y \geq -16$. The function f is negative for $-4 < x < 4$. The domain of $\sqrt{x^2 - 16}$ is $(-\infty, -4] \cup [4, \infty)$, while its range is $[0, \infty)$.

> **18.0 Students determine whether a relation defined by a graph, a set of ordered pairs, or a symbolic expression is a function and justify the conclusion.**

Problem: Does the equation $x^2 + y^2 = 1$ determine y as a function of x? Explain.
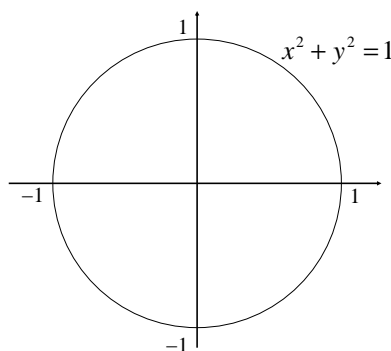
> **Solution:** Given a point (x, y) in the Cartesian plane, its distance to the origin is given by the formula
>
> $$D = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}.$$
>
> By taking square roots, the equation $x^2 + y^2 = 1$ is equivalent to
>
> $$\sqrt{x^2 + y^2} = 1.$$
>
> Therefore the equation represents the set of points (x, y) in the plane whose distance from the origin is 1, that is, the unit circle.
>
>
>
> We can see that to each x, $-1 < x < 1$, there corresponds two values of y that satisfy the equation. This means that the equation $x^2 + y^2 = 1$ does not define y as a function of x.

Answer: The equation $x^2 + y^2 = 1$ does not define y as a function of x, because to each value of x, $-1 < x < 1$, there corresponds two values of y.

> **Further Explanation:** One can also answer this question by attempting to solve for y as follows. Since $x^2 + y^2 = 1$, we can solve and get $y^2 = 1 - x^2$. Taking the square root of both sides, and using the fact that $\sqrt{y^2} = |y|$, we get
>
> $$|y| = \sqrt{1 - x^2} \quad \Rightarrow \quad y = \pm\sqrt{1 - x^2}.$$
>
> This equation shows that each value of x gives rise to two values of y, (except for $x = 1$ or $x = -1$). Finally, notice that the graph of the equation fails to pass the *Vertical Line Test*, which is a graphical way to determine whether or not a relation is a function.

20.0 Students use the quadratic formula to find the roots of a second-degree polynomial and to solve quadratic equations.

Problem: Solve for x: $2x^2 - 3x - 5 = 0$.

Solution: If $ax^2 + bx + c = 0$, $a = 0$, then the quadratic formula states that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

are the solutions of the equation. In this case, $a = 2$, $b = -3$ and $c = -5$. Thus

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9 + 40}}{4}$$

$$= \frac{3 \pm 7}{4}.$$

The $\pm$ indicates that we obtain two solutions by first considering the sum in the numerator and then the difference:

$$x = \frac{3 + 7}{4} = \frac{10}{4} = \frac{5}{2}$$

$$x = \frac{3 - 7}{4} = \frac{-4}{4} = -1.$$

Answer: The solutions are $x = \dfrac{5}{2}$ and $x = -1$.

20.0 Students use the quadratic formula to find the roots of a second-degree polynomial and to solve quadratic equations.

Problem: Let $f(x) = ax^2 + bx + c$. Suppose that $b^2 - 4ac > 0$. Use the quadratic formula to show that f has two roots.

Solution: We may assume here that $a = 0$, or else we have a linear function. In general, the roots of a polynomial $f(x)$ are the values of x for which $f(x) = 0$. (These will also represent the x values where the graph of the function crosses the x-axis.) Setting $f(x) = 0$ is equivalent to setting

$$ax^2 + bx + c = 0,$$

which by the quadratic formula has solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Notice that if $D = b^2 - 4ac > 0$, then the square root in the quadratic formula will be a real number. (That is, $\sqrt{D}$ exists and is a real number.) The $\pm$ sign in the equation then yields two distinct solutions of the quadratic equation, namely:

$$x = \frac{-b + \sqrt{D}}{2a},$$
$$x = \frac{-b - \sqrt{D}}{2a}.$$

Answer: Since $D = b^2 - 4ac > 0$, we have $\sqrt{D} > 0$, so there are two real roots.

Further Explanation: The expression $D = b^2 - 4ac$ in the quadratic formula is known as the *discriminant*. In light of the fact that a quadratic equation can be solved with the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, which features $\sqrt{D}$, the value of D immediately gives us information about the roots of the function $f(x) = ax^2 + bx + c$. In particular, if $D > 0$, then as noted in the solution above, f has two (distinct) real roots. If $D = 0$, then $x = -\frac{b}{2a}$ is the only solution, and we'd say f has a *double root*. In the case that $D < 0$, the square root of D is *purely imaginary* and the roots of f are *complex numbers*. That is, $\sqrt{D} = di$, for some positive real number d, where i satisfies $i^2 = -1$. In this case, we have $x = \dfrac{-b \pm di}{2a}$.

22.0 Students use the quadratic formula or factoring techniques or both to determine whether the graph of a quadratic function will intersect the x-axis in zero, one, or two points.

Problem: At how many points does the graph of $g(x) = 2x^2 - x + 1$ intersect the x-axis?

Solution: The graph of g intersects the x-axis at the x-values for which $g(x) = 0$. Setting $g(x) = 0$ yields
$$2x^2 - x + 1 = 0.$$

If we use the quadratic formula to solve for x, we obtain

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-1) \pm \sqrt{1 - 4(2)(1)}}{2(2)}$$
$$= \frac{1 \pm \sqrt{-7}}{4}.$$

Since $\sqrt{-7}$ is not a real number, the function g has no real roots, and thus the graph of g does not intersect the x-axis.

Answer: The graph of g does not intersect the x-axis.

Further Explanation: One may also complete the square to convert the quadratic function into *vertex form* $(y = a(x - h)^2 + k)$. We can write

$$g(x) = 2\left(x^2 - \frac{1}{2}x\right) + 1 = 2\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) + 1 - \frac{1}{8}$$
$$= 2\left(x - \frac{1}{4}\right)^2 + \frac{7}{8}.$$

This shows that the graph of g is a parabola with vertex $(1/4, 7/8)$ that opens upward, which means that the graph of g does not cross the x-axis.

23.0 Students apply quadratic equations to physical problems, such as the motion of an object under the force of gravity.

Problem: A ball is launched from the ground straight up into the air at a rate of 64 feet per second. Its height h above the ground (in feet) after t seconds is $h = 64t - 16t^2$. How high is the ball after 1 second? When is the ball 64 feet high? For what values of t is $h = 0$? What events do these represent in the flight of the ball? (Adapted from CERT 1997, 21.)

Solution: Let $h(t) = 64t - 16t^2$. If we set $t = 1$ in the equation for the height, we see that
$$h(1) = 64(1) - 16(1^2) = 48,$$
so that the ball is at a height of 48 feet after 1 second.
The ball is 64 feet high when $h(t) = 64$, that is, when

$$64t - 16t^2 = 64$$
$$-16t^2 + 64t - 64 = 0.$$

Dividing each term of the equation by -16 yields $t^2 - 4t + 4 = 0$, which can be factored to obtain $(t - 2)^2 = 0$. Thus, when $t = 2$, $h(t) = 64$, so that after 2 seconds the height is 64 feet.
Setting $h(t) = 0$ yields

$$64t - 16t^2 = 0 \quad \Rightarrow \quad 4t - t^2 = 0 \quad \Rightarrow \quad t(4 - t) = 0.$$

Thus $h(t) = 0$ when $t = 0$ or $t = 4$. Physically, these values represent that the height of the ball is 0. That is, the ball is at ground level when $t = 0$, which is the instant the ball is launched, and $t = 4$, which is 4 seconds after the ball is launched.

Answer: After 1 second, the ball is 48 feet up; after 2 seconds it is 64 feet up. The ball is at ground level when initially launched at 0 seconds, and after 4 seconds, when it finishes its flight.

23.0 Students apply quadratic equations to physical problems, such as the motion of an object under the force of gravity.

Problem: The braking distance of a car (how far it travels after the brakes are applied until it comes to a stop) is proportional to the square of its speed. Write a formula expressing this relationship and explain the meaning of each term in the formula. If a car traveling at 50 mph has a braking distance of 105 feet, then what would its braking distance be if it were traveling at 60 mph? (ICAS 1997, 6)3.

Solution: Let b represent the braking distance in feet, and let s represent its speed in mph. Then the first statement of the problem describes the relationship

$$b = ks^2,$$

where k is a constant of proportionality. We can solve for k since we are given values for b and s that satisfy this relationship:

$$105\,\text{ft} = k(50\,\text{mph})^2 \quad \Rightarrow \quad k = \frac{105\,\text{ft}}{2500\,\text{mi}^2/\,\text{hr}^2} = \frac{21\,\text{ft} \cdot \text{hr}^2}{500\,\text{mi}^2}.$$

The units of k may look strange here, but as long as b has units feet and s has units miles per hour, all the units will work out in the end. If the car travels at 60 mph, then

$$\begin{aligned} b &= \frac{21\,\text{ft} \cdot \text{hr}^2}{500\,\text{mi}^2}(60\,\text{mi}/\,\text{hr})^2 \\ &= \frac{21\,\text{ft} \cdot \text{hr}^2 \times 3600\,\text{mi}^2/\,\text{hr}^2}{500\,\text{mi}^2} \\ &= 151.2\,\text{ft}. \end{aligned}$$

Therefore, it takes 151.2 ft for the car to stop when traveling at 60 mph.

Answer: The breaking distance is 151.2 ft.

> **24.0 Students use and know simple aspects of a logical argument:**
>
> **24.1 Students explain the difference between inductive and deductive reasoning and identify and provide examples of each.**
>
> **24.2 Students identify the hypothesis and conclusion in logical deduction.**
>
> **24.3 Students use counterexamples to show that an assertion is false and recognize that a single counterexample is sufficient to refute an assertion.**

Problem: Provide numbers to show how the following statement can be false and if possible describe when it is true: $\sqrt{a^2 + b^2} < a + b$ whenever $a \geq 0$ and $b \geq 0$. (Adapted from CERT 1997, 39).

> **Solution:** Notice that if $a = b = 0$, then $\sqrt{0^2 + 0^2} = 0 = 0 + 0$, so that the original statement is false. The statement is also false when $a = 0$ and $b = 1$, or when $a = 1$ and $b = 0$, and in general if at least one of a or b is zero. The statement is true for $a = 1$ and $b = 1$, $a = 1$ and $b = 2$, and $a = 3$ and $b = 1$.
>
> In fact, much more can be said. If both a and b are greater than zero, then
>
> $$a^2 + b^2 < a^2 + b^2 + 2ab = (a + b)^2,$$
>
> since $2ab > 0$. In this case, we can take the square root on either side and obtain
>
> $$\sqrt{a^2 + b^2} < \sqrt{(a + b)^2} = |a + b| = a + b,$$
>
> proving the result is true precisely when both a and b are positive.

Answer: The statement is true whenever a and b are both positive.

> **Further Explanation:** If we have a right triangle for which a and b are the lengths of the legs and c is the length of the hypotenuse, then the statement is equivalent to the statement $c < a + b$.

> **25.0 Students use properties of the number system to judge the validity of results, to justify each step of a procedure, and to prove or disprove statements:**
>
> **25.1 Students use properties of numbers to construct simple, valid arguments (direct and indirect) for, or formulate counterexamples to, claimed assertions.**
>
> **25.2 Students judge the validity of an argument according to whether the properties of the real number system and the order of operations have been applied correctly at each step.**
>
> **25.3 Given a specific algebraic statement involving linear, quadratic, or absolute value expressions or equations or inequalities, students determine whether the statement is true sometimes, always, or never.**

Problem: Suppose that 9 is a factor of xy, where x and y are counting numbers. At least one of the following is true. Which of the following statements are <u>necessarily</u> true? Explain why.

1. 9 must be a factor of x or of y.

2. 3 must be a factor of x or of y.

3. 3 must be a factor of x and of y.

(CERT 1999, 89.)

> **Solution:** A condition B is said to be *necessary for* A if A implies B. In other words, A cannot be true unless B is also true. Let A be the statement "9 is a factor of the product xy." We investigate whether the following candidates for B are necessary conditions for A.
>
> 1. If $x = 6$ and $y = 15$, then 9 is a factor of the product $xy = 90$, but 9 is neither a factor of x nor of y. This statement is not necessarily true.
>
> 2. If 9 is a factor of xy, then 3 is a factor of xy, since $9 = 3^2$. If 3 is a factor of xy, then since 3 is prime, 3 must divide one of x or y. This statement is necessarily true.
>
> 3. If $x = 9$ and $y = 2$, then 9 is a factor of $xy = 18$, but 3 is not a factor of y. This statement is not necessarily true.

Answer: Statement 2 is the only statement that must necessarily be true.

25.0 Students use properties of the number system to judge the validity of results, to justify each step of a procedure, and to prove or disprove statements:

25.1 Students use properties of numbers to construct simple, valid arguments (direct and indirect) for, or formulate counterexamples to, claimed assertions.

25.2 Students judge the validity of an argument according to whether the properties of the real number system and the order of operations have been applied correctly at each step.

25.3 Given a specific algebraic statement involving linear, quadratic, or absolute value expressions or equations or inequalities, students determine whether the statement is true sometimes, always, or never.

Problem: A problem is given, to find all solutions to the equation $(2x+4)^2 = (x+1)^2$. Comment on any errors in the following proposed solutions:

Original equation:	$(2x+4)^2 = (x+1)^2$
Take the square root of both sides:	$2x+4 = x+1$
Subtract x and 4 from both sides to obtain:	$2x+4-x-4 = x+1-x-4$
Simplify to conclude:	$x = -3$

Solution: Recalling that $\sqrt{a^2} = |a|$ for a real number a, we see that taking the square root of both sides of the equation yields

$$|2x+4| = |x+1|.$$

This is where the error in the solution occurs.

If we continue solving the initial problem, we obtain four equations:

(i) $\quad 2x+4 = x+1$ $\qquad$ (ii) $\quad -(2x+4) = x+1$

(iii) $\quad 2x+4 = -(x+1)$ $\quad$ (iv) $\quad -(2x+4) = -(x+1)$

Notice that equations (i) and (iv) are equivalent, and yield $x = -3$ as found above. Equations (ii) and (iii) are also equivalent, but yield a different solution:

$$2x+4 = -(x+1) \quad \Rightarrow \quad 2x+4 = -x-1 \quad \Rightarrow \quad 3x = -5,$$

or $x = -5/3$. Therefore, the initial solution is incomplete since the solutions are $x = -3$ and $x = -5/3$.

Answer: The first step of the solution is incorrect. The correct answer is $x = -3$ and $x = -5/3$.